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Instructional-based learning of cyclic quadrilateral theorems: Making geometric thinking visible and enhancing learners' spatial and geometry cognitions

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Geometry learning has a long history with the connection to human cognitive development. The ability to mentally or physically orient 2D shapes or 3D objects in space is believed to support achievement in science, technology, engineering, and mathematics (STEM) disciplines. Thus, geometry by its description is characterised by space; hence its conceptualisation requires deep spatial and geometry sense which can be developed through instructional-based learning (IBL). While the current study supports the evidence of educational technology tools in support for effective teaching and learning, it explores IBL as an alternative source of enhancing learners' spatial and geometry cognitions and making geometric thinking visible to learners in the classroom. The experimental design research applied a mixed method to collect data from randomly selected 50 Grade 11 Mathematics learners from one of the high schools in the Cetshwayo district, South Africa. Quantitative data were collected through a pre-test-post-test design and analysed using independent sample t-test whereas qualitative data were collected using content analysis and analysed using descriptive analysis. The analysis was guided by the theoretical frameworks of Van Hiele theory of levels of geometric understanding and Zazkis et al.'s visualisation-analysis model. Based on the data set collected, the review of literature, and the theoretical frameworks, the ramification is that IBL has a positive effect on learners' spatial and geometric cognitions and makes geometry thinking visible to learners in the classroom.

Contribution: The recommendation is to explore the effect of hybrid learning design, thus the integration of IBL and dynamic geometry environments on the development of learners' spatial and geometry cognitions.

Keywords: achievement; cyclic quadrilateral theorems; geometric thinking; instructional-based learning; spatial skills.

Introduction

Learners' geometric thinking and understanding are measured by their achievement. As such, this not only hinders learners from growing as creative and critical thinkers, but also deprives teachers of information needed to enrich teaching and identify learners' challenges in learning. Accordingly, Garet et al. (2001) mentioned that learning is less meaningful if the learning process cannot be seen, activated, and observed. Thus, learners need to understand the processes of learning, how reasoning and knowing take place, and also the sort of influences or modifications that can occur in these processes. According to Hattie et al. (2016), guiding learners to know when and how thinking is happening is crucial to their cognitive development and also promotes a positive attitude toward learning. Drawing from Garet et al. (2001) and Hattie et al.'s (2016) assertions, making geometry thinking visible in learning processes enables learners to have control of their understandings and encourages active participation.

Researchers (Booth et al., 2013; Ng & Lee, 2009) in their various scholarly works hypothesise that visual representations (such as graphs, models, number lines, and geometric figures) are means of making geometry thinking visible. This is based on the assumption that visual representations communicate geometric ideas and, hence, support creative and critical reasoning (Driscoll et al., 2016). Other research groups have brought forward strong evidence against the physical or mental orientation of geometric objects in space to support geometry understanding (Strong &

Note: Additional supporting information may be found in the online version of this article as Online Appendix 1 and Online Appendix 2.

Smith, 2001; Stylianides et al., 2016). As such, what can be drawn from Strong and Smith (2001), Stylianides et al. (2016) and Driscoll et al.'s (2016) assertion is that geometric thinking becomes visible to learners when learning design incorporates visual representations. Furthermore, from the abovementioned and the pioneering work of Polya (1957), there still exists a positive relationship between classrooms' instructional-based learning (IBL) which supports learners' spatial and geometric thinking.

Thus, while prior studies focus on learners' success in geometric proving (Lazonder & Harmsen, 2016; Ngirishi & Bansilal, 2019), enhancing teachers' pedagogical content knowledge (Ndlovu, 2014; Ndlovu et al., 2013), and problem-solving skills and achievement (Battista et al., 2018), less is known about developing the cognitive factors such as spatial skills and geometric reasoning responsible for learning geometry. Furthermore, the recent surge in educational technological tools (such as MATLAB, CAS, and many more) designed to support the facilitation of effective teaching and learning is limited to South African rural high schools and other developing countries. Resources such as computers, internet, and energy are not readily available to all high schools, particularly in developing countries. In addition, learning Euclidean geometry becomes too abstract for learners when they cannot observe learning processes (Garet et al., 2001). Thus, the essence of making the geometric learning process visible is crucial to learners' cognitive development and also promotes a positive attitude toward learning.

Hence, as an alternative, the current study applies IBL as opposed to educational technologies as an immediate source that supports spatial reasoning and geometric reasoning visible to learners. Instructional-based learning is a pedagogical approach that employs a systematic task in developing learners' critical thinking, creative thinking, and geometric reasoning (Changwong et al., 2018; Schmaltz et al., 2017). Geometry is the study of the relationships and properties of objects in space and hence requires spatial sense in problem-solving and proving (Battista et al., 2018; Fujita, 2012). Furthermore, learning geometry has a long history of enhancing mathematics reasoning due to its inherent involvement with space. For instance, in their pioneer scholarly work, Linn and Petersen (1985) established that the ability to physically or mentally orient geometric objects in space is a wielded cognitive tool for learning. Consequently, this ability is pertinent to learners' visual interactions with learning activities in the classroom. In support of this, Polya (1957) mentioned that learners explore and assess their understanding before and after interacting with learning activities.

The reintroduction of Euclidean geometry as a compulsory section in Mathematics Paper 2 of the Grade 12 National Senior Certificate (NSC) examination after it was made optional for a certain group of learners is targeted to produce learners who are creative or critical thinkers and problem-solvers. Euclidean geometry carries about 50 ± 3 percentage points of the total of 150 marks of the Mathematics Paper 2 of the NSC; however, learners perform below the average pass rate of 45% on highly cognitive demanding questions (Umalusi, 2014). This poor performance of learners in Euclidean geometry contributes to the poor pass rates in Mathematics. Studies probing learners' poor performance in Euclidean geometry mainly focus on teachers' pedagogic content knowledge and learners' problem-solving skills, and achievement rather than the cognitive factors such as spatial skills and geometric reasoning responsible for learning Euclidean geometry.

The essence of making geometry thinking visible encompasses learners' ability to develop their understandings through active participation, creative and critical thinking. Hence, the implication is that, while IBL is applied by several studies (Damopolii et al., 2019; Driscoll et al., 2016; Strong & Smith, 2001; Stylianides et al., 2016) as a pedagogical tool for enriching classroom learning, its effect on making geometry thinking visible to learners and developing learners' spatial and geometry cognitions is less known; hence the discourse of the current study. Based on the introduction of the study, the research questions under consideration are: (1) *What is the effect of IBL of cyclic quadrilateral theorems on the development of learners' spatial and geometric cognitions?* and (2) *What is the effect of IBL on making geometric thinking visible to learners in the classroom?*

Literature review

Euclidean geometry deals with the study of geometric shapes based on different axioms and theorems. Thus, geometry relationship with space provides convincing evidence that geometry learning taps deep into human cognitive development. Additionally, other researchers such as De Villiers and Heideman (2014) and Naidoo and Kapofu (2020) further argued learning geometry provides a conceptual framework for developing mathematical reasoning. Chambers (2008) mentioned that geometry is a field of Mathematics that offers the 'enormous potential of bringing the subject alive' (p. 187). Hence, based on this assumption made by Chambers (2008), many educational curriculums are hinged on geometry worldwide (Hanna, 2000). For instance, the South African curriculum has gone through drastic changes reintroducing geometry into the mainstream after it was made optional for certain categories of learners from 2008 to 2013 (De Villiers & Heideman, 2014). With this change, the South African curriculum supports the development of learners who are creative and critical thinkers and problem-solvers (Naidoo & Kapofu, 2020; Ngirishi & Bansilal, 2019). However, the reintroduction of geometry into mainstream (i.e., Mathematics Paper 2) adds to the already crumbling pass rate in Mathematics.

Several studies have alluded that learners' poor performance in Euclidean geometry is a result of teachers' limited content knowledge and poor pedagogical approach (De Villiers & Heideman, 2014; Mouton et al., 2012; Naidoo & Kapofu, 2020;

Ngirishi & Bansilal, 2019). Sadly, while ongoing content workshops are organised to develop teachers' content knowledge (Ndlovu et al., 2013) less is done to capacitate them with pedagogical knowledge in making geometry thinking visible in the classroom. Ritchhart and Church (2020) asserted that making thinking visible encompasses deep learning. Ritchhart and Church further explained deep learning as the conceptual understanding of content coupled with the ability to think critically in problem-solving. According to Naidoo and Kapofu (2020), thinking becomes visible when learners are able to create their own understanding during the learning process which fosters active participation.

Ndlovu (2014) asserted that many teachers adhere to more conventional methods of teaching, such as teaching pedagogy which is teacher-centred, with less emphasis on prior knowledge and experience, which makes geometry learning too abstract. In response to this instructional conventionalism, for learners to develop deep geometry understanding, the reintroduction of Euclidean geometry in South Africa's Curriculum and Assessment Policy Statement (CAPS) requires an instructional approach to be more conceptually based with less attention to rote learning (Ndlovu, 2014). Hence, geometry learning becomes less meaningful when learners cannot see the processes of learning, how thinking and knowing take place. Naidoo and Kapofu (2020) claimed that meaningful learning of Euclidean geometry happens when learners directly interact with learning activities. For example, Polya (1957) argued 'an element that we introduce in the hope that it will further the solution is called an auxiliary element' (p. 46). As incurred by Gridos et al. (2021), introducing auxiliary lines in geometric diagrams provides a clue to solving some geometric proof-related problems. Fan et al. (2017) furthered the notion by claiming that adding auxiliary lines allows learners to explore geometry ideas required in solving some challenging or high-level cognitive problems. As such, the addition of auxiliary lines to a geometry diagram allows learners to have direct physical interactions with the geometric diagram, which is crucial to human cognitive development (Jansen et al., 2013; Stylianides et al., 2016). Hence, based on the aforementioned, it is worth examining if spatial skills and geometry thinking are dependent on learners' physical interaction with geometric diagrams.

Visualisation and conceptualisation processes

A substantial amount of research has proven that spatial-visualisation skills play a vital role in conceptualisation processes in science, technology, engineering, and mathematics (STEM) and other disciplines (Casey et al., 2015; Gilligan et al., 2017; Ho & Lowrie, 2014). As promising is the claim is that learners who perform in these disciplines tend to show high spatial abilities (Gilligan et al., 2017; Ho & Lowrie, 2014; Stylianides et al., 2016).

Despite these studies, there is still evidence from the work of McLeay (2006) that suggests that there is a link between imagery, spatial skills, and problem-solving. Others such as Newcombe (2010) are of the view that illustrations or pictures support spatial thinking. Newcombe and Shipley (2014) attempted to advance their view by evaluating visual and spatial reasoning for design creativity with mixed results. Despite the studies conducted by Newcombe (2010), McLeay (2006), and Newcombe and Shipley (2014), a contrasting view obtained from Nguyen and Rebello (2011) is that learners encounter challenges when confronted with multiple representations with particular reference to image movements of 3D objects, alternatively manipulation in the mind. The question worth pursuing is how to make the task of representation given other representations and how changes in one representation affect others in problem-solving situations. Thus, the essence of image movements of 3D objects or orientation in the mind is identified as spatial-visualisation (Strong & Smith, 2001), but has proven to be difficult to comprehend following a number of candidate reasons.

Research has demonstrated a connection between learners' achievement, cognitive reasoning, and visual-spatial skills. However, it is unclear whether this relationship accounts for interactive visualisation (Casey et al., 2015; Gilligan et al., 2017; Ngirishi & Bansilal, 2019). In response to this inconclusive report, Lim et al. (2020) for example examined interactive visualisations with students of 15 and 16 years old in learning calculus. Lim et al.'s findings show visualisation enhances the learners' understanding. Earlier on, Phillips et al. (2010) in evaluating 40 articles on visualisations in mathematics classrooms found that visualisation may interfere when the goal is an analytic skill.

Theoretical framework

The current study applies the Van Hiele (1999) geometric theory of understanding and the visual and analytic (VA) model of Zazkis et al. (1996) addressing the phenomena under consideration, namely 'instructional-based learning of cyclic quadrilateral theorems: means through making geometric thinking visible and enhancing learners' spatial and geometry cognitions'. Thus, Van Hiele's (1999) theory of geometric learning provides the pedagogical approach to learning geometry whereas Zazkis et al.'s (1996) VA model outlines the visual and analytical interaction during problem-solving processes.

According to Abdullah and Zakaria (2013), high school geometry is somewhat integrated with algebra and, as such, Van Hiele's (1999) theory may not solely be suitable for analysing learners' understanding of Euclidean geometry. Due in part to this drawback, Zazkis et al.'s (1996) VA model has been proven to account for the coordination between spatial-visual and analytical aspects in problem-solving situations and conceptualised didactic approach (such as IBL) that supports spatial-visual skills and analytic cognition. Thus, while the two theoretical frameworks differ in context and purpose, their combined effects may provide a new dimension in developing learners' spatial and geometry cognitive thinking.

Van Hiele's theory of geometry thinking

Van Hiele's (1986, 1999) theory of geometry thinking outlines the pedagogical approach to learning geometry characterised by two main components, namely the definitions and descriptors of the cognitive levels in which learners progress in learning geometry (Abdullah & Zakaria, 2013; Fujita & Jones, 2007). In addition, Van Hiele (1986) further explains the sequential orders for learning geometry that must be adhered to succeed in proof-oriented geometry, namely Visualisation, Analytic, Abstract, Deduction, and Rigour (Fujita & Jones, 2007; Van Hiele, 1999). Additionally, the theory is influenced by conceptual understanding of geometric properties and analysis. Hence, a brief description of Van Hiele's (1986) levels of understanding of geometry is discussed as follows:

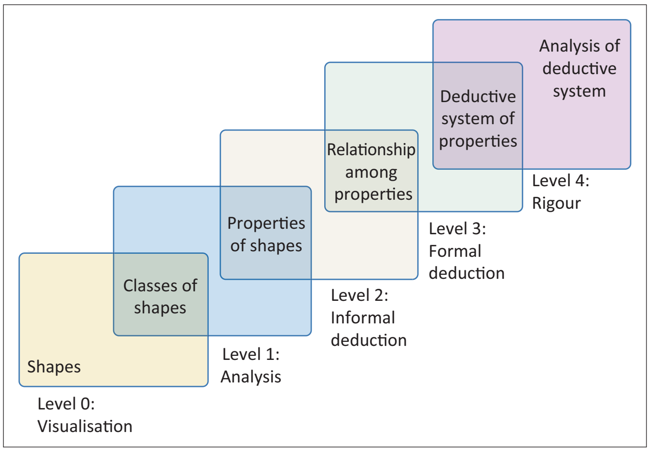
- Level 0 (Visualisation): Recognising geometric figures by their shapes.
  Learners at the level of visualisation think about geometric shapes and differentiate between their similarities and differences by holistically using shapes. For instance, the learner sees a circle as any geometric shape that looks like the sun, and a rectangle as a shape like a table. Hence, the learner disagrees with a square as a rectangle since a rectangle has two pairs of opposite sides equal whereas a square has all sides equal. As such, the learner's judgement is by the holistic look of the geometric shapes.
- Level 1 (Analysis): Analysing geometric shapes and naming properties.
  At the level of analysis, Van Hiele (1986) explains that geometric thinking involves identifying and classifying relations among properties. As such, at this level, learners can analyse properties; however, they are unable to prearrange geometrical properties. The idea is that the learner learns the properties based on common features in isolation of each other. That is, the learners still cannot see that a square is a rectangle since they cannot connect their properties.
- Level 2 (Abstraction): Perceive the relationships between properties and figures.
  This level shows the learner has mastered level 1 and they can move to the level of noticing the relationship among the properties and being able to make informal deductions between shapes and their properties. Thus, at this level, learners can distinguish between geometric shapes using their properties. Henceforth, at this level, the learner accepts a square as a rectangle and is able to distinguish not all rectangles are squares (i.e, geometric properties relations). In addition, ideas are constructed on geometrical thinking (Duval, 2017); hence learners are able to make a deduction. For instance, learners at this level are able to make the conjecture that since an isosceles triangle is symmetrical, the base angles are the same. This also implies the learner understands definitions and can make a valid statement; however, they do not know their functions in geometric proofs. Furthermore, drawing from Van Hiele's (1986) geometric thinking model, it can be argued that geometric thinking and analytical thinking get richer from one level to another level which sheds light on Zazkis et al.'s (1996) VA model.
- Level 3 (Deduction): Give a deductive proof.
  This level requires learners to be able to authenticate the relationships between geometric shapes by using deductive axiomatic proofs. Thus, learners reason and make formal deductions between properties of geometric shapes making use of axioms, definitions, theorems, corollaries, and postulates. In addition, learners can work with abstract statements using geometric properties to make logical conclusions.
- Level 4 (Rigour): Understand the way mathematical systems are established.
  At this level, learners rigorously compare different axioms when thinking of geometric shapes. Accordingly, learners must conceptually understand definitions, proof, and axioms and apply these definitions and axioms in making deductive reasoning. Furthermore, as hypothesised by Van Hiele (1999), learning levels of geometry are interconnected as illustrated in Figure 1.

As such, Van Hiele (1999) outlined that understanding geometry starts with spatial-visualisation even though the mechanism linking spatial skills and numerical sense is not explicitly explained. Hence, the study applied Zazkis et al.'s (1996) VA model to analyse the coordination between spatial-visualisation and analysis during problem-solving processes.

Visualisation-Analysis model

Physical manipulation or orientation in 2D or 3D is seen as a tool for enhancing problem-solving (Jansen et al., 2013; Zazkis et al., 1996). Garet et al. (2001) mentioned that visual representation makes geometric reasoning visual which supports conceptual understanding. Zazkis et al. (1996) furthered the argument by providing visual and analysis alteration during problem problem-solving process as follows.

According to this model of Zazkis et al. (1996), the process begins with an act of visualisation,$V1$ which could be any visual representation (drawing, image, graph, computer animation, or mental image). The model further explains that



Source: Van Hiele, P.M. (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics, 5*(6), 310–316. https://doi.org/10.5951/TCM.5.6.0310

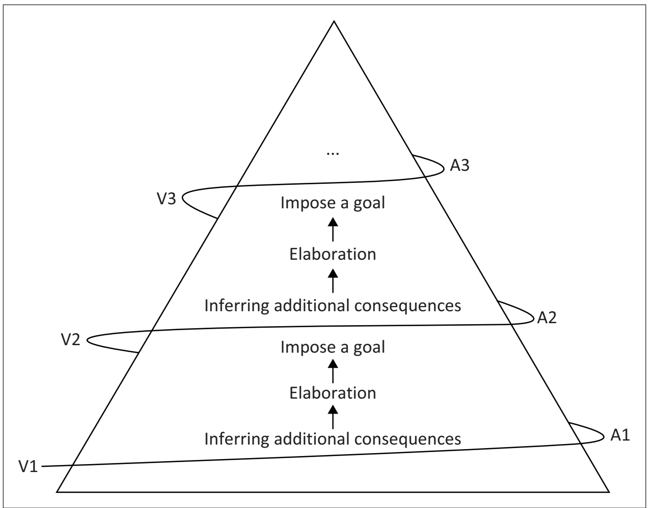
FIGURE 1: Van Hiele theory of geometric thinking.

the image is analysed during stage $A1$ which consists of some kind of coordination of the objects and processes constructed in step $V1$. This analysis may lead to new constructions. In a succeeding act of visualisation, $V2$, the learner returns to the same 'picture' object used in $V1$, but as a result of the analysis in $A1$, the picture may change. Thus, as the sequence is repeated, each act of analysis is dependent on the previous act of visualisation. Zazkis et al. (1996) concluded that spatial-visualisation and analysis movement are seen as distinct and different; however, as the coordination continues the two forms of thinking become synthesised at the top of the triangle (refer to Figure 2).

Methodology

This study explored the correlation between IBL cyclic quadrilateral theorems and learners' spatial and geometric cognitions as well as the effect of IBL making geometric thinking visible to learners. Guided by these objectives, the literature review, and the theoretical framework, the current study is based on a classroom setting in which teaching and learning materials (TLMs) are readily available. The study, through IBL, adopts the learning design that incorporates classroom hands-on activities and an investigative approach in learning cyclic quadrilateral theorems as one of the key conditions to be met for the experimental study.

A random sampling technique through experimental design (Creswell, 2021) was utilised in selecting a sample size of 50 Grade 11 Mathematics learners from one of the high schools in the King Cetshwayo district in South Africa. The sample size was randomised into two equal groups: the experimental group (N = 25) and the control group (N = 25). Prior to the study, the two groups were introduced to the cyclic quadrilateral theorem to ensure homogeneity among the groups in terms of their prior knowledge. Afterwards, a baseline assessment test on Euclidean geometry (specifically cyclic quadrilaterals theorems) was administered to the two groups. The average mean scores were compared to ensure



Source: Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group D4. *Journal for Research in Mathematics Education, 27*(4), 435–457. https://doi.org/10.5951/jresematheduc.27.4.0435

FIGURE 2: Visualisation-Analysis model.

both the experimental and the control groups had the same baseline. Purdue Spatial Visualisation Tests: Visualisation of Rotation (PSVT/R) was further administered to the two groups which served as a baseline test before the two teaching methods were applied.

Instructional-based learning was employed as an instructional tool in teaching and learning cyclic quadrilateral theorems for the experimental group. The IBL learning approach is learner-centred; thus, the experimental learners follow an investigative approach in proving or solving geometry problems and theorems (refer to Online Appendix 2).

Meanwhile, the conventional method was applied in teaching and learning for the control group. Conventional learning pedagogy is teacher-centred, with less emphasis on prior knowledge and experience. Then after, a post-test (achievement test) was administered, the results were analysed using an independent sample t-test (refer to Table 2). The PSVT/R was again administered to the groups to examine the effect of the two teaching methods on the development of learners' spatial and geometric cognitions. Descriptive data were collected through content analysis to examine whether or not there is a correlation between IBL and making geometric thinking visible in the classroom. Thus, two samples of learners' worksheets were randomly drawn from the experimental and control groups before and after applying the two different teaching methods and analysed descriptively. The analysis was based on learners' ability to solve related cyclic quadrilateral tasks and proofs, ranging from lower to higher-order cognitive questions. The evaluation is based on learners' logical reasoning, critical thinking, spatial ability, and geometric reasoning in solving higher-order questions. In addition, making geometric reasoning visible to learners is measured by their ability to provide correct geometric reasons to support the problem solutions given.

Instrumentation and data collection

Data were collected using three instruments: achievement tests (i.e. pre-test and post-test), PSVT/R test, and content analysis. Firstly, a pre-test based on related problems on cyclic quadrilateral theorems and proofs was administered to ensure the two groups had similar prior knowledge. Afterward, two different teaching methods were employed in teaching and learning cyclic quadrilateral theorems. The experimental group was taught using an IBL approach. Learning design was integrated with classroom hands-on practical and investigative approach learning of cyclic quadrilateral theorems which lasted for a week.

For the first day, the experimental learners were taught the usage of mathematical tools such as compass, protractor, and ruler in construction and measuring triangles in a circle. On the second day, the learners constructed a circle and inscribed triangles to validate the conjecture: opposite interior angles of a cyclic quadrilateral are supplementary by following the instructions (refer to Figure 1-OA2 in Online Appendix 2).

On the third and fourth day, the same practical and investigative approaches were used to investigate the relationship between the exterior angle of a cyclic quadrilateral and its interior opposite angle (refer to Figure 2-OA2 in Online Appendix 2).

As such, the experimental learners having gone through the activities of pencil-paper drawing and experimentation are able to visually interact with the geometric diagrams. Also, IBL lessons promoted active participation and peer discussions, enhancing learners' critical thinking.

In contrast, the control group was taught using a conventional approach (i.e. a teaching approach that is more teacher dominated and does not involve giving the learners any exploratory activities); the lessons lasted for the same timeframe. Thus, cyclic quadrilateral theorems were explained and their accompanying diagrams were sketched on the board followed by related worked examples, and exercises were given. After, the two teaching methods were applied post tests were administered and data were collected and analysed.

Validity and reliability of data collection instrument

Prior to the study, the data collection instruments were tested on the targeted group (i.e. Grade 11 Mathematics learners) from the same high school. Also, experienced mathematics teachers were consulted to proofread and assess the questions which enabled the authors to refine the questions based on the time allocation, curbing ambiguity in the questions' language, ordering, and cognitive levels. This was done to ensure the validity and reliability of the data collection instruments (Creswell, 2021).

In addition, the reliability coefficient (KR-20) was calculated for each question on the geometric cyclic quadrilateral theorems research tasks and the difficulty was evaluated according to the result in Table 1 (Baykul, 2000; Isman & Eskicumali, 2003). The questions that scored in the range $0.30 \leq \text{KR-20} \leq 0.50$ are included and the mean score of the items is 3.5 which is considered to be reliable for Grade 11 Mathematics learners.

Discussion of results

An independent sample *t*-test was carried out to determine whether there was significant difference in prior knowledge of cyclic quadrilateral theorems between the two groups and the results are illustrated in Table 2.

Referring to Table 2, the mean score of the control is 13.68, and the experimental group score 13.76. The mean difference between the two groups was 0.08 with a *t*-value of 0.51. However, the two-tailed significance (i.e. *P*-value) was 0.96 ($P > 0.05$) which means there is no significant difference in the mean score among the groups. Thus, the two groups have similar prior knowledge of cyclic quadrilateral theorems.

A post-test was then administered after applying IBL as a treatment in the experimental group and the conventional method in the control group. An independent sample *t*-test was again carried out to analyse the effects of the two different teaching methods in terms of developing the learners' spatial and geometric cognition and as well as making geometric thinking visible to learners. The results were analysed in Table 3.

Referring to Table 3, the mean score of the control group was 21.72 while the experimental group obtained 29.08. The mean difference between the groups was 7.36, and the *t*-value was 4.13. The *P*-value was 0.00 ($P < 0.05$), which means there is a significant difference between the mean scores of the two groups. Hence, the experimental group outperformed the counterpart in the post-test after applying IBL as a pedagogical tool in teaching and learning cyclic quadrilateral theorems.

Furthermore, based on the research evidence available, visual interaction with geometric diagrams enhances both spatial and geometric cognition; hence, the experimental learners were tested on PSVT/R before and after applying IBL in teaching and learning cyclic quadrilateral theorems. An independent sample *t*-test was carried out before and after applying IBL to determine whether or not there was a statistical difference in the mean scores of the experimental group and the results were analysed in Table 4.

The result in Table 4 shows that the control group obtained a mean score of 13.68, and the experimental group 5.68. The mean difference between the groups was 0.08, with a *t*-value of 0.051. Meanwhile, the *P*-value was 0.96 ($P > 0.05$), which means there was no significant difference in the mean scores between the two groups. Hence, the control and experimental groups had similar spatial-visualisation skills, before the two teaching methods were applied. After the interventions (i.e. applying IBL) a post-test on PSVT/R was again administered to both groups, and the results were analysed in Table 5.

Referring to Table 5, the control group obtained a mean score of 11.08 and 13.80. The mean difference between the two groups was 2.72, and the *t*-test value was 4.27. However, the *P*-value of the groups was 0.00 ($P < 0.05$). This indicates a significant difference in the mean scores between the two groups. Therefore, IBL lessons considerably affected the experimental learners' spatial skills and geometric thinking. A paired sample *t*-test was further carried out to compare the pre-test and post-test results of the control and experimental groups and presented in Table 6.

The results in Table 6 show the experimental group obtained a mean score of 15.54 and a *P*-value of 0.00 ($P < 0.00$), indicating there is a statistical difference between the pre-test and the post-test. The control group obtained a mean score of 12.52 and a *P*-value of 0.00 ($P < 0.00$), which means there was a statistical difference between the pre-test and post-test after applying the two teaching methods.

TABLE 1: Difficulty levels of items.

Difficulty of items	Assessment of item
0.70–1.00	Too easy
0.50–0.69	Easy
0.39–0.49	Intermediate difficulty
≤ 0.29	Too difficult

Source: Baykul, Y. (2000). *Egitimde ve Psikolojide Olcme: Klasik ve Test Teorisi ve Uygulamasi.* and OSYM Yayinlari and Isman, A., & Eskicumali, A. (2003). *Egitimde Planlama ve Degerlendirme*. Degisim Yayinlari

Note that PSVT/R is a standard test for testing spatial-visualisation skills.

TABLE 2: Results of the independent *t*-test on the pre-test of both groups before the two different methods were applied (N = 25).

Group	Mean	SD	Mean difference	T*	Significance (two-tailed)
Control	13.68	5.41	-0.080	-0.051	0.96
Experimental	13.76	5.68	-	-	-

SD, standard deviation.

*, T-value significant at P < 0.05.

TABLE 3: Results of independent *t*-test on the post-test of both groups (N = 25).

Group	Mean	SD	Mean difference	T*	Significance (two-tailed)
Control	21.72	5.28772	-7.360	-4.13	0.001
Experimental	29.08	7.17589	-	-	-

SD, standard deviation.

*, T-value significant at P < 0.05.

TABLE 4: Results of the independent *t*-test on the pre-test for spatial ability for both groups before the two teaching methods were applied (N = 25).

Group	Mean	SD	Mean difference	T*	Significance (two-tailed)
Control	13.68	5.41	-0.080	-0.051	0.96
Experimental	13.76	5.68	-	-	-

SD, standard deviation.

*, T-value significant at P < 0.05.

TABLE 5: Results of independent *t*-test on the post-test for spatial ability for the two groups after the two teaching methods were applied (N = 25).

Group	Mean	SD	Mean difference	T*	Significance (two-tailed)
Control group	11.08	2.37	-2.72	-4.27	0.00
Experimental group	13.80	2.12	-	-	-

SD, standard deviation.

*, T-value significant at P < 0.05.

TABLE 6: Results of the paired independent samples test of the groups (N = 25).

Pair	Group	Mean	SD	T	Significance (two-tailed)
1	Control group Pre-test, Post-test	12.52	5.05	-5.86	0.00
2	Experimental Pre-test, Post-test	15.54	4.44	-	0.00

SD, standard deviation.

Content analysis

There are six questions in all ranging from low to higher order in terms of cognitive levels. Note that questions 1, 2, 3, and 6 were analysed since questions 4 and 5 are of the same cognitive level with 3 and 6. Two scripts were randomly selected from both groups and analysed descriptively. Thus, through the solution presented, the learners' logical thinking, critical thinking, spatial, and geometric thinking were analysed descriptively. Hence, making geometric thinking visible to learners in the classroom was evaluated through their ability to recall geometric facts, and theorems, and apply logical reasoning in providing problem solutions. The analysis was guided by the theoretical frameworks of Van Hiele's (1999) order of geometric thinking and Zazkis et al.'s (1996) VA model. These frameworks outline the pedagogical approach to learning geometry and the visual and analysis interactions that take place in problem-solving situations.

The analysis starts with the production of the question followed by the solution and the reason given by the learner. In answering Question 1, the learner needs to recall that the two radii *OM* and *OP* produce equal base angles hence $\hat{O}_1 = 96°$ and $\hat{T} = 48°$. Hence, to solve for N the learner has the alternatives of applying the theorems 'opposite < *s* of $\hat{T}$ and $\hat{N}$ are supplementary' or 'angle subtended by $\hat{O}_2 = \hat{N}$ (i.e. angle at the centre is twice the circumference)'. This schema (i.e. schema 1) was used to analyse the solution presented by the learner (C22) from the control group before the conventional teaching method was applied (refer to Figure 3-OA2 in Online Appendix 2) and VA (refer to Table 1-OA2 in Online Appendix 2).

Thus, referring to the solution provided by the learner (C22) to Question 1, he was able to visualise and recall two radii produced equal base angles, hence, the angle $\hat{O}_1 = 96°$. However, he failed to apply the theorem of the angle subtended from the centre is twice the angle subtended at the circumference. Also, the learner failed to provide a correct geometric reason to support his solution. For instance, comparing the learner's third VA, he made the error of stating $\hat{O}_1 = \hat{N}$ and providing the reason as '$\hat{O}_1$ and $\hat{N}$ reflex angle' (refer to Table 1-OA2 in Online Appendix 2). This may be as a result of the learner's understanding being based on memorisation; thus, he does not conceptually understand the geometric concepts. According to Van Hiele's (1999) theory of levels of geometry thinking, the learner is only working between levels 1 and 2 (i.e. analysis and abstraction).

Schema 1 was again used to analyse the solution presented by the learner (E11) from the experimental group before the IBL teaching approach was applied.

Referring to the solution provided by (E11) to Question 1 (refer to Figure 4-OA2: in Online Appendix 2) and VA (refer to Table 2-OA2 in Online Appendix 2), the learner was able to visualise and recall that two radii produce equal base angles, hence the sum of the base angles; $M\hat{O}P = 96°$; however, he failed to apply the theorem that the angle subtended from the centre is twice the angle subtended at the circumference. Also, the learner failed to provide a correct geometric reason to support his solution. Furthermore, the learner failed to recall the cyclic quadrilateral theorem (i.e. $\hat{N} + \hat{T} = 180°$). As such, the learner's inability to recall geometric facts and theorems in providing the solution to Question 1 may be attributed to inappropriate pedagogical approach: his understanding is built on rote learning.

Also, in providing solutions to Question 2, the learner needs to understand the geometric sign or language (refer to Figure 5-OA2 in Online Appendix 2 and Table 3-OA2 on

Online Appendix 2): the line segment DC equals EC means ΔCDE is an isosceles triangle, hence it has the same base angles. Furthermore, the learner needs to recall the tangent-chord theorem of ΔCDE, thus, $x = E_2$ the opposite angle of the exterior. Also, ED//DC means line *ED* and *DC* are parallel, which requires the learner to apply the properties of parallel lines, hence $E_2 = C_2$ are alternative angles. This schema (i.e. Schema 2) was applied in analysing the solution presented by the learner C8. As such, referring to the solution presented by the learner (refer to Figure 5-OA2 in Online Appendix 2) and VA (refer to Table 3-OA2 in Online Appendix 2), from the control group, the learner was able to recall and apply the tangent-chord theorem (i.e. the exterior angle is equal to the opposite interior angle, $\hat{E}_2 = x$). Also, the learner understands the geometric language, which is that ΔCDE is an isosceles, hence has equal base angles (i.e. $\hat{D}_2 = \hat{E}_2$). However, the learner failed to apply the properties of transactional lines in identifying the third angle. As hypothesised by Van Hiele (1999), learning geometry starts with visualisation, the lower level, before proceeding to a higher level. Furthermore, Polya (1957) mentioned that learners' ability to scrutinise geometric diagrams gives them a clue about the problem solution. As such, what can be drawn from Polya's (1957) and Van Hiele's (1999) assertion is that the learners' failure to interact with the geometric diagram could be the reason they failed to identify the third angle. Hence, it may be said that the emphasis on conventional teaching and learning of the cyclic quadrilateral theorem only contributes to rote learning which ultimately hinders learners' problem-solving abilities.

Schema in 2 was again applied in analysing the solution presented by learner E15. Thus, referring to the solution (refer to Figure 6-OA2 in Online Appendix 2) and VA (refer to Table 4-OA2 in Online Appendix 2), presented by the learner (E15) from the experimental group, he was able to recall and apply the tan chord theorem and the properties of transactional lines in identifying other angles that are equal to the angle $A\hat{D}F$ (i.e. x). However, the learner failed to identify the third angle which is also equal to $A\hat{D}F$. Thus, the learner failed to scrutinise the geometric diagram and identify the type of the inscribed triangle given or perhaps he did not understand the geometric language that identified that line, $BE = CD$, hence ΔCDE is an isosceles triangle (i.e. same base angles). In responding to this, Van Hiele (1999) explains that learners need to grasp a lower level of geometry learning before proceeding to a higher level. Furthermore, Polya (1957) explains that learners' ability to scrutinise geometric diagrams helps them to explore geometric facts and theorems required in problem-solving situations. Hence, it can be drawn from Polya's (1957) and Van Hiele's (1986) assertions that the learner's understanding is based on rote learning and also their failure to interact with the geometric diagram could be the reason they failed to identify the third angle.

In providing solution to Question 6 (refer to Online Appendix 1), the learner has to apply the property of straight lines and apply either the theorem 'opposite interior of quadrilateral is supplementary' or 'exterior angle equal to the opposite interior angle' to prove *ABCD* is a cyclic quadrilateral. This schema (i.e. Schema 3) was used in analysing the solution presented by learner C24 from the control group.

Referring to the solution (refer to Figure 7-OA2 in Online Appendix 2) and VA (refer to Table 5-OA2 in Online Appendix 2) presented by the learner (C19), they were able to visualise that *BAE* and *BCF* produce straight lines, hence they were able to apply the properties of a straight line in solving for the angles A_2 and C_2. However, the learner failed to apply any of the theorems involving the cyclic quadrilateral in proving *ABCD* is a cyclic quadrilateral.

Schema 3 was again applied in analysing the solution presented by the learner (E 4) before IBL was applied (refer to Figure 8-OA2 in Online Appendix 2) and VA (refer to Table 6-OA2 in Appendix 2). This refers to the solution and VA presented by the learner (E4) from the experimental group after IBL was applied in teaching and learning cyclic geometry. The learner was able to recall the geometric fact that the angle on a straight line is 180° and they calculated the value of $\hat{A}_2$ and C_2 which sums to 180°. In addition, the learner understood the converse theorem of cyclic quadrilaterals, hence making the conclusion that the opposite interior angles of a cyclic quadrilateral sum to 180°. In this regard, it can be said the learner has a deep understanding of geometric concepts. According to Damopolii et al. (2019), the IBL approach to teaching geometry allows learners to develop their own understanding which leads to long-term memory. Naidoo and Kapofu (2020) asserted that geometric thinking becomes visible when learners can create their own understanding during the learning process which enables them to solve higher-order questions. In support of this, Ritchhart and Church (2020) asserted that making thinking visible encompasses deep learning. Thus, conceptual understanding of content coupled with the ability to think critically promotes problem-solving abilities. Hence, in conclusion, it was observed that the interactive nature of IBL enhanced learners' critical thinking which was a key to problem-solving skills.

Table 7 compares the percentage score per question in the post-test between the control and experimental group after the two teaching methods were applied in teaching and learning cyclic quadrilateral theorems.

The achievement test (i.e., the post-test) consists of six questions (refer to Online Appendix 1) structured in order of lower to higher cognitive levels which examine the learner's ability to visualise, recall, and apply geometric ideas and theorems in problem-solving situations and proving. Figure 3 shows the analysis of the percentage per question scored by the control and experimental groups.

The line graph (see Figure 3) gives the analysis of the groups' performances per question after applying the two different

TABLE 7: Comparing the percentage score per question in the post-test between the control and experimental groups.

Item	Descriptions of problem solution	Percentage score per question of the control group	Percentage score per question of the experimental group
Question 1	Recall geometric facts and theorem in providing problem solutions.	76	94
Question 2	Recall and integrate geometric facts and theorems in providing problem solutions.	72	92
Question 3	Recall and integrate geometric facts and analytical reasoning in providing problem solutions.	70	90
Question 4	Recall and integrate geometric facts and analytical reasoning.	66	88†
Question 5	Recall and integrate geometric facts and analytical reasoning in proving the converse of cyclic quadrilateral theorem.	40	80
Question 6	Require geometric thinking, spatial skills, creative and critical thinking (i.e. adding of auxiliary lines) coupled with analytical thinking in proving.	41	84

†, $\frac{44}{50}$.

teaching methods in learning cyclic quadrilateral theorems. Question 1 requires learners' ability to recall geometric facts and theorem to solve the geometric problem. Referring to Table 7, the control group scored 78% whereas the experimental group 94% for the question. For Question 2 (refer to Online Appendix 1), the control group scored 72% in providing a correct answer and correct geometric reason for the answer whereas the experimental group scored 92%. Question three and four (refer to Online Appendix 1) are classified as a level three type of question which requires the learners to recall and integrate geometric ideas, facts, and theorem in providing the problem solution. Hence, the control group scores 68 and 40 percent whereas the experimental group scores 88 and 80 percent respectively. Thus, on average the experimental group outperformed the control group in terms of their ability to recall and integrate geometric facts, ideas, and theorems in problem-solving.

Question 5 (refer to Online Appendix 1) requires the learners to recall and integrate geometric facts and analytical reasoning in proving the converse of cyclic quadrilateral theorem. The control group scored 40% and in the experimental group 80% was able to provide the correct proof. Question 6 (refer to Online Appendix 1) is a level 4 question that require the learner's ability the recall and apply geometric thinking, spatial skills, creative, critical thinking (i.e. adding of auxiliary lines) and analytical thinking in providing the problem solution. Overall, the control group scored 40% whereas the experimental group scored 84%. Overall, the experimental learners performed better than the control group.

Interpretation of results

The study set out to explore the research questions: 1) What is the effect of IBL of cyclic quadrilateral theorems on the development learners' spatial and geometric cognitions? 2) What is the effect of IBL on making geometric thinking visible to learners in the classroom?

Research question 1

As a recap, a baseline test (specifically on cyclic quadrilateral theorems) was administered to the control and experimental groups. However, the experimental group was further given the standard spatial test as a baseline. Table 2 shows the analysis of the results of the pre-test: the control group learners scored a mean value of 13.68 and the experimental group learners scored 13.76. The two-tailed significance (i.e. *P*-value) was 0.96 ($P > 0.05$) which shows there was no significant difference between the two groups. After, the two groups were given lessons on cyclic quadrilateral theorems using different teaching approaches. Thus, the experimental group was taught using IBL which incorporates a practical investigative approach in teaching and learning cyclic quadrilateral theorems whereas the control group was taught by conventional method (i.e. learning pedagogy which is teacher-centred, with less emphasis on prior knowledge and experience). Hence, referring to the analysis of the results of the achievement test (i.e. post-test) in Table 3, the experimental group obtained a mean score of 21.72 while the experimental group obtained 29.08. The *P*-value was 0.00 ($P < 0.05$) indicating there is a significant difference that favours the experimental group.

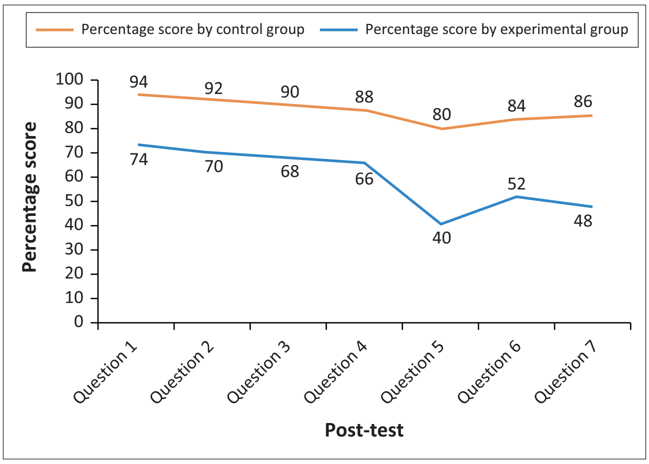


FIGURE 3: Analysis of the percentage score per question between control and experimental groups for the post-test.

Hence, the assumption is the activities of concrete manipulation, pencil-paper drawing and experimenting were utilised to facilitate learners' visual interaction with the geometric diagrams. According to Polya (1957), learners' direct interaction with learning materials facilitates the learning process. Thus, through IBL, paper-pencil drawing and experimentation the learners explore geometric properties, develop logical reasoning and creative and critical thinking required for problem-solving skills and proving (Jones, 2000). Also, the activities of using the practical investigative approach in proving cyclic quadrilateral theorems afforded the experimental learners the skills to develop their own conceptual understanding leading to long-term memory. The study's findings support those of Damopolii et al. (2019), who found that learners' visual interaction with geometric diagrams—for example, the addition of auxiliary lines to geometry—improves their spatial reasoning and achievement. Zazkis et al. (1996) mentioned that there is coordination between visual and analysis and they complement each other in problem-solving processes.

In support of this, the analysis of the results in Table 7 shows overall the experimental group outperformed by far their counterparts based on higher cognitive questions (i.e. 4, 5, and 6; refer to Online Appendix 1) that require learners' logical reasoning, creative and critical thinking, spatial and geometric thinking. Komatsu and Jones (2020) opined that learners' ability to use mathematical tools not only supported spatial and geometric thinking but also enhanced effective learning and participation. Hence, the assumption is that experimental learners develop conceptual understanding through IBL of cyclic quadrilateral theorems. Accordingly, Van Hiele's (1999) theory of levels of understanding geometry explicitly explains that learners having gone through the sequential order can rigorously compare different axioms and use these definitions and axioms in making deductive reasoning. Hence, through the practical investigative learning approach of learning cyclic quadrilateral theorems, the experimental learners were able to establish the processes of learning and how thinking and knowing take place, and, as such, geometric thinking becomes visible.

Research question 2

Geometry by its definition implicates space, hence learning geometry requires human spatial and geometric reasoning for its conceptualisation. According to Kilgo and White (2014), visual representation communicates mathematical ideas. Kilgo and White (2014) further explain learners' classroom hands-on activities such as paper-and-pencil drawing and experimenting support spatial and geometric cognition. However, Ritchhart and Church (2020) suggested that visual representations make geometric thinking visible to learners. Hence, Table 4 gives the analysis of the results before and after applying IBL. The experimental group obtained a mean score of 13.68 and 13.76 before and after IBL was applied and the *P*-value of the groups was 0.00 ($P < 0.05$). This indicates there is a significant difference between before and after IBL was applied in teaching and learning cyclic quadrilateral theorem. Furthermore, comparing the VA (refer to Online Appendix 2 Figure 6-OA2) presented by the learner (E18) from the experimental group and the learner (C24) from the control group (refer to Table 5-OA2 and Table 6-OA2 in Online Appendix 2); the experimental learner (E18) exhibits deep geometry understanding. Thus, they provided correct geometric reasoning to back the solution whereas the control learner (C24) failed to provide correct geometric reasoning to support their solution.

As such, the result clearly distinguishes IBL as an effective pedagogical tool for teaching and learning cyclic quadrilateral theorems compared to the conventional approach of learning. Hence, what can be drawn from this is that the interactive nature of IBL is the mechanism responsible for enhancing the experimental learners' conceptual understanding. Thus, IBL supports deep learning. Ritchhart and Church (2020) opined that visible thinking is characterised by deep learning. Furthermore, it can also be deduced from the VA (refer to Table 3-OA2, Online Appendix 2), there is coordination between the experimental learner's visualisation and numerical skills, as hypothesised by Zazkis et al. (1996). Hence, the idea is classroom hands-on activities such as paper-pencil drawing of circles and experimentation and adding auxiliary lines to explore geometric ideas make geometry thinking visible to learners which arouses curiosity, and promotes a positive attitude toward learning. Hattie et al. (2016) mentioned that making geometric thinking visible in the classroom enables learners to have control of their own understandings and it enhances active participation.

Conclusion

This study, unlike other prior research cited in this article, establishes that active exploration (e.g. classroom hands-on activities and paper-and-pencil construction and experimenting) facilitates geometric proving and also supports active learning. Additionally, the findings demonstrate that learners' physical interaction (e.g. adding auxiliary lines to geometric diagrams) is a tool for developing spatial and geometry cognition. Unlike previous studies too, this current study establishes that IBL encourages learners' active participation and critical thinking.

More importantly, the evidence established that the interactive nature of IBL makes geometry thinking visible and as such develops deep understanding, leading to long-term memory. This means that through IBL lessons, the learners gain a constructive understanding of geometric ideas or axioms required in solving high-order geometric problems. We further argued and established that classroom hands-on activities (i.e. paper-pencil drawing and adding auxiliary lines to geometric diagrams) enable learners to actively explore geometric ideas, which are crucial for geometric proving.

Recommendation

The current study supports the evidence that educational tools promote effective teaching and learning; however, as an alternative IBL is employed as a tool for developing learners' spatial and geometry cognitions. Furthermore, it is established through learners' visual interaction with geometric diagrams that geometric thinking is visible which enhances their deep understanding of cyclic quadrilateral theorems.

As such, the finding recommends learning geometry should incorporate visual representation which makes geometric thinking visible and communicates geometric reasoning. Thus, IBL should be seen as an immediate source that enhances learners' spatial and geometric cognition and should be extended to even higher institutions. Finally, one foremost recommendation is to integrate both IBL and dynamic geometric environments in the future to explore the effect of cyclic quadrilateral theorems on the development of learners' spatial skills and geometric cognition.

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Competing interests

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Authors' contributions

R.G. and G.A. conceptualised the study and article, invoked applied methodology, and performed the formal analysis. G.A. carried out the investigation and curated the data. R.G. and G.A. prepared the draft manuscript and through several iterations of review and editing produced the final article for submission.

Ethical considerations

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Data availability

The data that support the findings of this study are openly available from the authors, R.G. and G.A., upon reasonable request.

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